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Fractality and the percolation transition in complex networks

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ABSTRACT

In this work we study the percolation phase transition of a family of scale-free networks previously introduced to explain the emergence of fractality and self-similarity in complex networks. This model introduces a parameter, e, that allows for tuning the level of fractality in the network. When $e=0$ the network is a pure fractal, when $e=1$ it is a pure small-world network. We examine link percolation for several intermediate values of e ($0 \leqslant e \leqslant 1$), and find that the transition follows a continuous function of e that converges to exactly 0 when the network exhibits the small-world property.

1. Introduction

We would like to dedicate this work to Prof. Morton Denn, who has made so many contributions to the field of complex and disordered systems, in honor of his 70th birthday.

Percolation in Complex Networks, such as the World Wide Web or networks of protein interaction, has recently attracted much attention since it allows for a study of the vulnerability or robustness of the connectivity under random or intentional breakdown of fractions of the network (Albert et al., 2000; Albert and Barabási, 2002; Newman, 2003; Dorogovtsev and Mendes, 2002). Percolation has also been applied to study the spread of epidemics in a population and to determine the fraction of a population that must be immunized for a disease not to spread. Many of these studies were done for the case of random scale-free networks (Callaway et al., 2000; Cohen et al., 2000, 2002; Gallos et al., 2005), in which random connections are made between nodes with the constraint of achieving a power-law degree distribution (Molloy and Reed, 1995) (the distribution of the number of links coming out of a node follows a power law $P(k) \sim k^{-\gamma}$) since many networks in Nature follow such distribution. These studies show that scale-free networks with $\gamma < 3$ (as typically occurs in real-world networks such as the Internet, the World Wide Web or biological networks of protein interactions) are robust under random removal, and if $\gamma > 4$ they exhibit a global breakdown of the network under removal of a fraction of its links or nodes.

Although random scale-free networks are indubitably important for these studies, they present several limitations since it fails to answer questions such as: What are the structural properties of the network (such as assortativity, degree correlations, average path length, degree distribution, etc.) that determine and characterizes its robustness? Is there a topological attribute that allows for tuning the resilience under removal of the network links or nodes? Much work has been devoted to these questions proposing model to study how the percolation transition and its critical exponents are affected by different topological properties. Furthermore, these models show that the degree distribution is not the only determinant factor for the robustness of a network, and, more specifically, degree correlations and the average path length are also important (Berker and Ostlund, 1979; Rozenfeld and ben-Avraham, 2007; Angeles Serrano and Boguna, 2006a, b; Kaufman and Andelman, 1984; Hong, 1984; Kaufman and Griffiths, 1981, 1984; Hinczewski and Berker, 2006).

In this work we show how *fractality* influences the percolation transition in complex networks. Few years ago, Song et al. (2005) found that many scale-free networks found in Nature are *fractal*. This indicates that, topologically, small parts of the network behave as the global structure. In addition, they developed an algorithm that models this behavior showing that the main property of fractal networks is that the hubs (the most connected nodes) get further apart as the network grows (Song et al., 2006). On the contrary, if the distance between hubs does not increase, the resulting network is a *small-world* in which the diameter (the longest shortest path) increases logarithmically with the size of the network.

The main ingredient of the model of Song et al. is the introduction of a parameter, e, that measures the probability that hubs remain connected as the network grows. Therefore, the parameter e can be

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understood as a level of fractality in the network. When $e=0$, the network is a pure fractal, and otherwise, when $e=1$ the network is a pure small-world network.

Section 2 describes the generalized version of the Song et al. network model (Gallos et al., 2007) and presents its main properties. Section 3 presents the percolation problem and studies the link percolation transition and its dependence with the fractality parameter e. Section 4 concludes this paper with a discussion of the results.

2. The network model

In this section we describe the algorithm to build the model network introduced by Song et al. and its basic topological properties. The network is constructed iteratively as follows (see Fig. 1): In generation $n=0$, start with two nodes connected by one link. Then, generation $n+1$ is obtained recursively by attaching m new nodes to the endpoints of each link l of generation n. In addition, with probability e we keep link l and add $x-1$ new links connecting pairs of new nodes attached to the endpoints of l (see Fig. 2). Otherwise, with probability $1-e$ we remove link l and add x new links connecting pairs of new nodes attached to the endpoints of l. For example, to obtain generation $n=1$ in Fig. 1 we connected two new nodes to the endpoints of the link of generation $n=0$. Since $e=0$, we remove the link of $n=0$. Finally, we add $x=2$ links between new nodes.

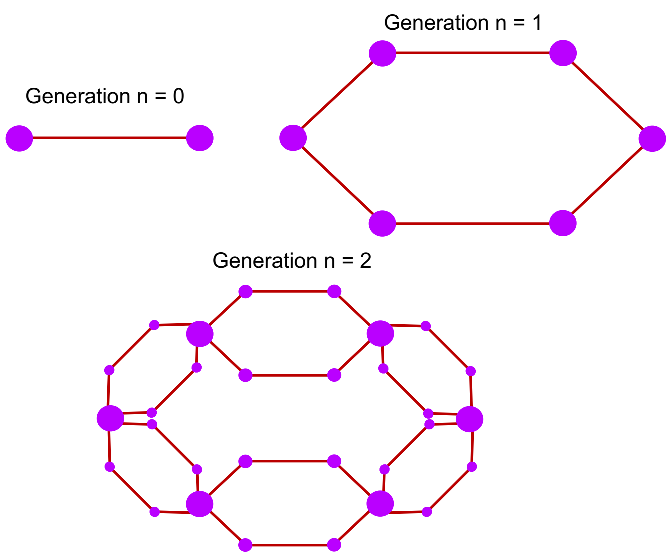

Fig. 1. Construction of a pure fractal network. Example of network model with parameters $n=0,1,2$; $m=2$; $x=2$; $e=0$.

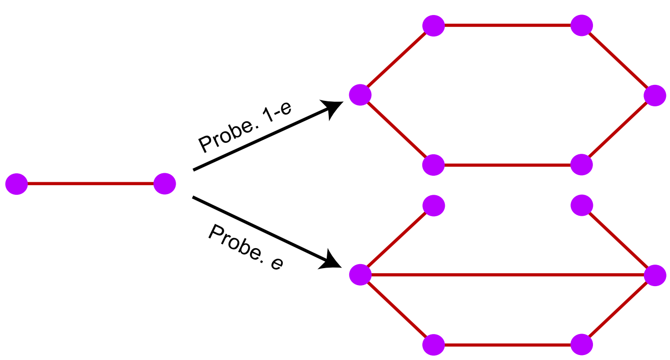

Fig. 2. Construction of network. With probability e the link between hub remains, otherwise, with probability $1-e$ it is replaced for another link between new nodes.

Notice that, for example, $x=1$ gives always a tree-structure (a loop-less network) regardless the value of e.

2.1. Topological properties

At generation n the number of links, M_n, increases by a factor $2m+x$, so that $M_n=(2m+x)^n$. On the other hand, each link in $n-1$ generates $2m$ nodes in generation n. Therefore, $N_n=2mM_{n-1}+N_{n-1}$, which admits the solution,

$$N_n=\frac{2m}{2m+x-1}(2m+x)^n+2-\frac{2m}{2m+x-1} \quad (1)$$

It has been shown (Song et al., 2006) that the degree distribution of the network model follows a power-law $P(k)\sim k^{-\gamma}$. The exponent $\gamma=1+\ln b/\ln s$ is determined by the scaling of the number of nodes, b, and the scaling of the degrees, s, as the network grows. In each generation the degree of a node increases by $s=m$ and from Eq. (1) we obtain $b=2m+x$. Therefore,

$$\gamma=1+\frac{\ln(2m+x)}{\ln m} \quad (2)$$

At the same time we can calculate the fractal dimension $d_f=\ln b/\ln a$, where $a=3-2e$ corresponds to the scaling of the distance as the network grows. Therefore,

$$d_f=\frac{\ln(2m+x)}{\ln(3-2e)} \quad (3)$$

It is important to notice that when $e=1$ the fractal dimension diverges, as expected, since the network becomes a pure small-world for which fractal dimensions are not trivially defined (Rozenfeld et al., 2007).

3. Percolation

The basic model of percolation is as follows (Kapitulnik et al., 1983; Stauffer and Aharony, 1991; ben-Avraham and Havlin, 2000). Consider a square lattice of $N\times N$ sites in which each bond (or link) is present with probability p, or otherwise absent with probability $q=1-p$. When p is small, there is a low concentration of bonds and the lattice is composed of small clusters, each consisting of a few sites (or nodes). As p increases, the size of clusters increases. Eventually, for some value of p a cluster that spans the lattice *from edge to edge* emerges. If the lattice is infinite, a spanning cluster appears at a well-defined critical threshold $p\equiv p_c$. In complex networks, since there is no notion of euclidean space or an evident embedding space (Rozenfeld et al., 2002), we study the emergence of a *giant component* (the component with the largest number of nodes) analogous to the spanning clusters in d-dimensional percolation. The giant component increases with the number of nodes in the network and emerges only if $p\geqslant p_c$. Moreover, if $p<p_c$ the network is formed by small components that grow slower than the number of nodes as the network grows.

In this section we study the link percolation transition in the model network presented above. We build a network with given n,m,x,e and proceed to remove each link with probability $1-p$, where $0\leqslant p\leqslant 1$. We then calculate the number of nodes in the largest component, P_∞, as p is changed. From classical percolation we know that P_∞ diverges as p approaches the percolation transition p_c. Below the transition ($p<p_c$), the network is made of small components that grow slower than the number of nodes in the network. Above the transition, in the limit $N\to\infty$, the network presents a *giant component* that grows proportionally to the number of nodes in the network.

We turn to the investigation of the percolation transition and its dependence with e. In Fig. 3 we study the behavior of P_∞ for different

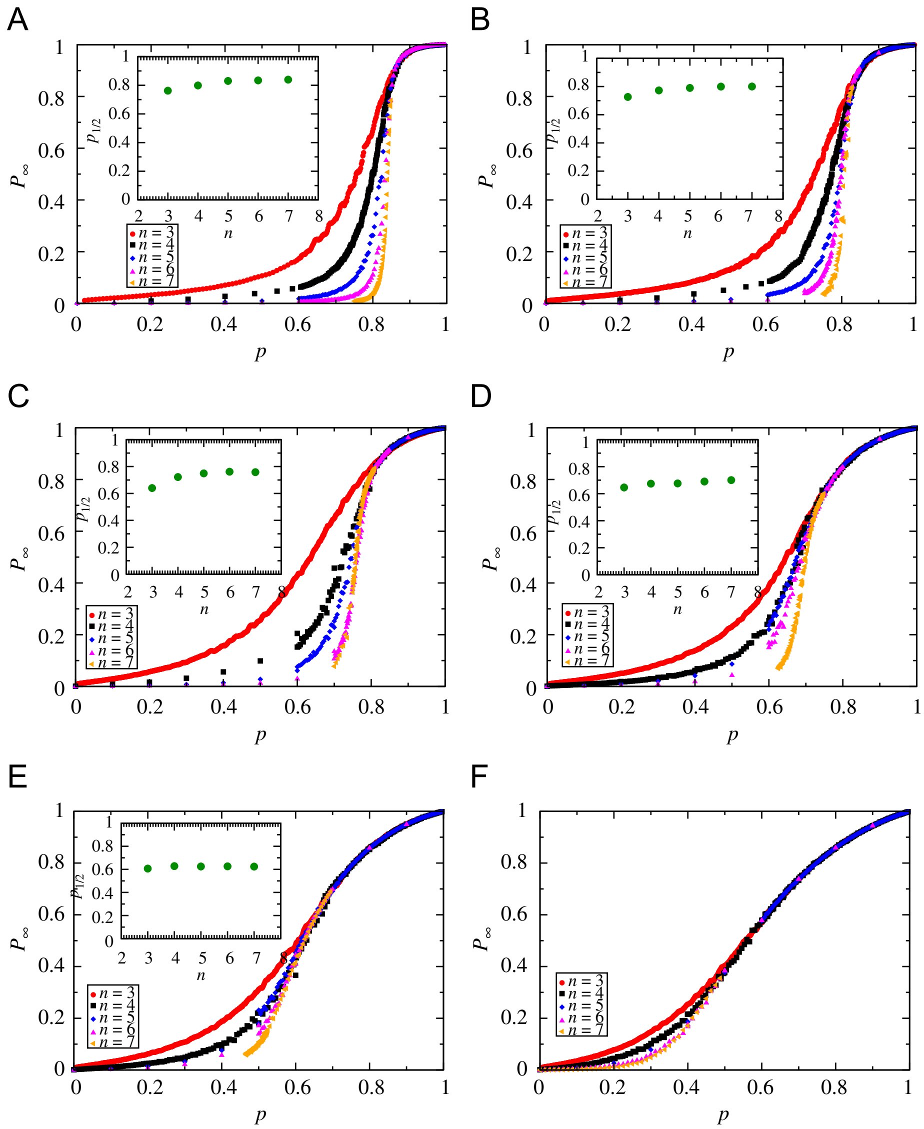

Fig. 3. P_∞ for the fractal scale-free model with $m=2$, $x=2$ and (A) $e=0$, (B) $e=0.2$, (C) $e=0.4$, (D) $e=0.6$, (E) $e=0.8$, (F) $e=1$. Each of these simulations are averaged over 1000 realizations. The insets show $p_{1/2}$, the value of p for which $P_\infty=\frac{1}{2}$, to estimate the percolation transition p_c.

values of e, when $m=2$, $x=2$ for different network sizes. We can see that as thenetwork grows, the percolation transition becomes more pronounced and converges to p_c, so that in the limit $N\to\infty$, P_∞ presents a jump at $p=p_c$. For example, in Fig. 3A, when $e=0$ (pure fractal) the transition is found at $p_c\approx 0.84$ (in agreement to what is reported in Rozenfeld and ben-Avraham, 2007). To estimate the value of p_c we find the value of p for which $P_\infty=\frac{1}{2}$ (Gallos et al., 2005). The insets in Fig. 3 show the estimation of p_c as the network grows. As e is increased and the network displays the small-world property, p_c decreases until values close to 0. In Fig. 3F we show P_∞ when $e=1$. We see that all curves take approximately the same value when $P_\infty=\frac{1}{2}$ so finite size analysis does not provide a conclusion regarding the percolation transition in the case of a small-world network model. Comparing with Rozenfeld and ben-Avraham (2007) we conclude that when $e=1$, the percolation transition must be at $p_c=0$ (as we confirmed through an analytical estimation), which implies that in order to reach a global breakdown of the network, one must remove a fraction 1 of the links.

In Fig. 4 we show the percolation transition as e is changed. We can see that p_c decreases uniformly and continuously until it reaches 0. We confirmed this result with the analytical calculation of the percolation transition (following the method of Rozenfeld and ben-Avraham, 2007) as e approaches 0.

4. Discussion

We have studied the percolation in a model network that was originally presented to explain the origin of fractality in complex networks. This model introduces a parameter e that can be thought as a “fractality tuner”. When $e=0$ we obtain a pure fractal and when $e=1$ the fractality breaks down and the network becomes small-world where distances grow logarithmically with the network size. We investigated the percolation phase transition through a Monte-Carlo simulation of the percolation process with network parameters $m=2$ and $x=2$ (although the choice of m and x is irrelevant for the behavior of p_c in terms of e, in accordance to what is found in Rozenfeld and ben-Avraham, 2007).

We find that p_c decreases continuously to 0 as the network goes from a pure fractal to a pure small-world network. This implies that as e is increased, the network becomes more resilient under removal

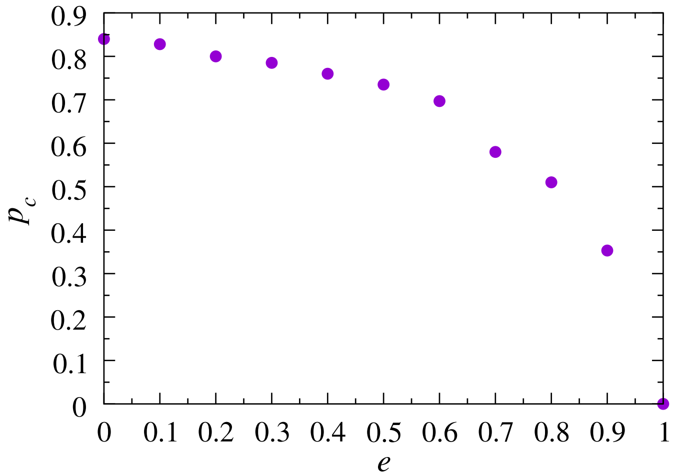

Fig. 4. The percolation transition p_c versus e for the model network with $m=2$ and $x=2$.

of its links. When $e=1$, and in the limit $N \to \infty$, in order to reach a global breakdown of the network, one needs to remove a fraction 1 of the links. That is, the small-world network is robust. For practical purposes, though, when $e=1$ and the network is finite ($N<\infty$), one may argue that the transition occurs for values larger than $p=0$. Indeed from Fig. 3F, we see that the number of nodes in the largest component reaches values close to 0 at $p \approx 0.2$.

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