Physica A 389 (2010) 3972–3977

Contents lists available at ScienceDirect

Physica A

journal homepage: www.elsevier.com/locate/physa

# From force distribution to average coordination number in frictional granular matter

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ARTICLE INFO

*Article history:*
Received 19 April 2010
Received in revised form 21 May 2010
Available online 8 June 2010

*Keywords:*
Force distribution
Granular matter

ABSTRACT

We study the joint probability distribution of normal and tangential frictional forces in jammed granular media, $P_\mu(f_t, f_n)$, for various values of the friction coefficient $\mu$, especially when $\mu = \infty$. A universal scaling law is found to collapse the data for $\mu = 0$ to $\infty$, demonstrating a link between the force distribution $P_\mu(f_t, f_n)$ and the average coordination number, $z_c^\mu$. The results determine $z_c^\mu$ for a finite friction coefficient, extending the constraint-counting argument of isostatic granular packing to finite frictional packings.


Granular matter undergoes a jamming transition evolving into an amorphous state with a non-zero yield stress as the density increases to a point where all particles are in contact [1]. It has been shown experimentally and numerically that the forces are inhomogeneously distributed within a jammed granular system, and they further appear to decay exponentially or stretch exponentially for large values of the force [2–5]. To date, there have been various theoretical attempts to describe the force distribution, predicting different behavior. For instance, lattice models and Boltzmann-equation approaches [5,6] predict an exponential decay, and an attempt to describe correlations in force transmission was accomplished in [7]. Attempts to fit experimental data within the energy ensemble [8] predict stretched exponential behavior. But the results are difficult to justify, since for granular matter energy is neither well defined nor conserved due to frictional forces [9]. An alternative approach is to use the so-called force canonical ensemble with a Boltzmann distribution where the boundary stress, not energy, is the conserved quantity [10–12]. Edwards' theory suggests a canonical ensemble [10] over external pressure, $\exp[-S - \Pi/A]$, where $\Pi$ is the stress boundary, $A = \partial\Pi/\partial S$ is the angoricity, analogous to temperature in equilibrium statistical mechanics, and $S$ is the entropy. It is of interest to reduce the above-defined force ensemble to obtain a single force distribution, but methods to accomplish this remain very difficult mainly due to the lack of knowledge on the density of states [11] (which can be calculated numerically [13]). A crude approximation would ignore correlations between forces and the contact network as well as the density of states, and would predict an exponential decay for the force distribution [5,10].

Besides the force distribution and the density of states, an additional quantity of interest in this study is the average coordination number, $z_c^\mu$, of a system at the jamming transition with interparticle friction coefficient $\mu$. Despite the importance of $z_c^\mu$ for determining the packing stability, there is only one theoretical framework to characterize $z_c^\mu$ related to the counting argument of the isostatic conjecture [14]. At the isostatic limit, the configuration of contact forces has a unique solution if the contact network is given, since the number of independent forces is identical to the number of balance equations. Previous works [4,15–20] have shown that packings at the jamming transition point are isostatic only for two extreme cases, $\mu = 0$ and $\mu = \infty$, with average coordination number $z_c^0 = 2D$ and $z_c^\infty = D + 1$ respectively, where $D$ is the number of dimensions. Recent studies [16] confirm that the indeterminacy of the force ensemble [17] reaches a minimum at $\mu = 0$ and $\infty$. Other studies would argue that isostaticity is only well defined at the frictionless point and not

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doi:10.1016/j.physa.2010.05.053

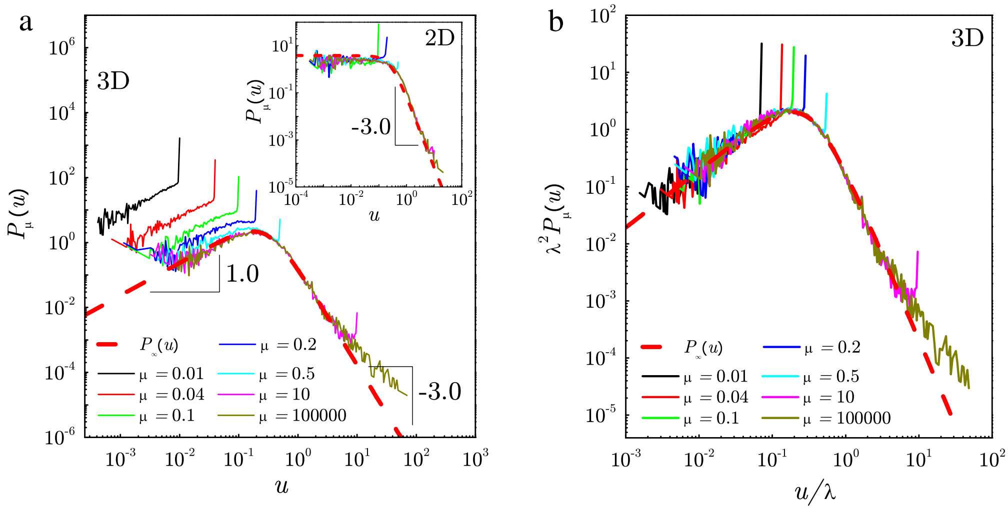


**Fig. 1.** (a) and the inset are log–log plots of the PDF of $u$ respectively in three dimensions and two dimensions for various values of $\mu$; (b) log–log plot of the collapsed $P_\mu(u)$ for various values of $\mu$ in three dimensions. The red dashed lines in (a) and (b) are plots of $P_\infty(u) = \frac{\kappa(D-1)}{(1+\kappa^2u^2)^{3/2}}\left(\frac{\kappa u}{\sqrt{1+\kappa^2u^2}}\right)^{D-2}$, where we use $\kappa = 3.80$ and 3.43, respectively, for two dimensions and three dimensions from a direct measurement of the simulation.

at $\mu = \infty$ [15]. Recent numerical studies in two dimensions show interesting power-law scaling approaching the isostatic limit as a function of friction [19].

Lacking more definite theoretical approaches to understand the force distribution, the density of states and $z_c^\mu$ for a general $\mu$, we perform a numerical study of the joint force distribution in frictional granular matter, $P_\mu(f_t, f_n)$. Here, the forces at the contacts are normalized by the average forces: in the tangential direction $f_t = \frac{F_t}{\langle F_t\rangle}$ and in the normal direction $f_n = \frac{F_n}{\langle F_n\rangle}$. We show that the key distribution is that of infinite $\mu$, interpreted in terms of the density of states and exponential statistics, providing guidance to theoretical attempts under the statistical framework. We show a universal form of the force ratio distribution $P_\mu(u)$, where $u$ is the ratio of normal and tangential force, $u = \frac{F_t}{F_n}$, valid for all $\mu$, and a scaling law is found to collapse all the $P_\mu(u)$ determining $z_c^\mu$ for packings. By using $P_\mu(u)$ we introduce a way to calculate the average coordination number for various values of $\mu$ based on the Maxwell construction of constraint arguments. Thus, we extend the isostatic condition from the limits of $\mu = 0$ and $\mu = \infty$ to finite $\mu$, providing the scaling of $z_c^\mu$, an unsolved nonlinear problem. Our results provide a connection between two important quantities to describe jammed matter: from force distribution to coordination number.

The packings we studied are composed of 10,000 equal-size spheres interacting with Hertz forces along the contact direction, $F_n$, and Mindlin forces in the tangential direction, $F_t$, plus the Coulomb condition, $F_t \le \mu F_n$ [18]. We first generate a gas state without friction at an initial volume fraction $\phi_i$ in a periodically repeated cubic box, then the packing is prepared with friction through a slow compression and relaxation process to achieve equilibrium at a given volume fraction and coordination number as close as possible to the limiting density of the jamming transition. The final macroscopic results do not depend on the details of this initial preparation since the system is still below the jamming condition, retaining the characteristics of a fluid. However, the ratio of normal to tangential interparticle force and its fluctuations strongly depend on the way a packing is constructed, as shown in [21]. A detailed description of the simulation is given in [20], and the computer code and packings are available at

We start by constructing an empirical formula for $P_\infty(f_t, f_n)$ based on two numerical results.

(i) We find that the ratio force distribution [2,4,18,19],

$$P_\mu(u) = \kappa \int_0^\infty f_n P_\mu(\kappa u f_n, f_n) \mathrm{d}f_n, \tag{1}$$

at infinite friction is characterized by two power laws with exponents equal to 0 and −3 in two dimensions, and 1 and −3 in three dimensions, respectively, at $u \to 0$ and $u \to \infty$, where $\kappa = \frac{\langle F_n\rangle}{\langle F_t\rangle}$ is an anisotropy parameter. Fig. 1(a) plots $P_\mu(u)$ for various values of $\mu$, showing that all $P_\mu(u)$ display similar behavior, having two power-law slopes except for a sharp peak at $u = \mu$, due to sliding contacts reaching the Coulomb threshold. A correct form of force distribution should predict this power-law behavior.

Notice that previous two-dimensional simulations [2,19] have reported a plateau of $P_\mu(u)$ in the region of $[0, \mu]$, corresponding to the first power law of $P_\mu(u)$ with the exponent equal to 0, shown in the inset of Fig. 1(a). The second power law only appears for very large values of $\mu$ and has not been reported by previous studies. We only show $P_\mu(u)$

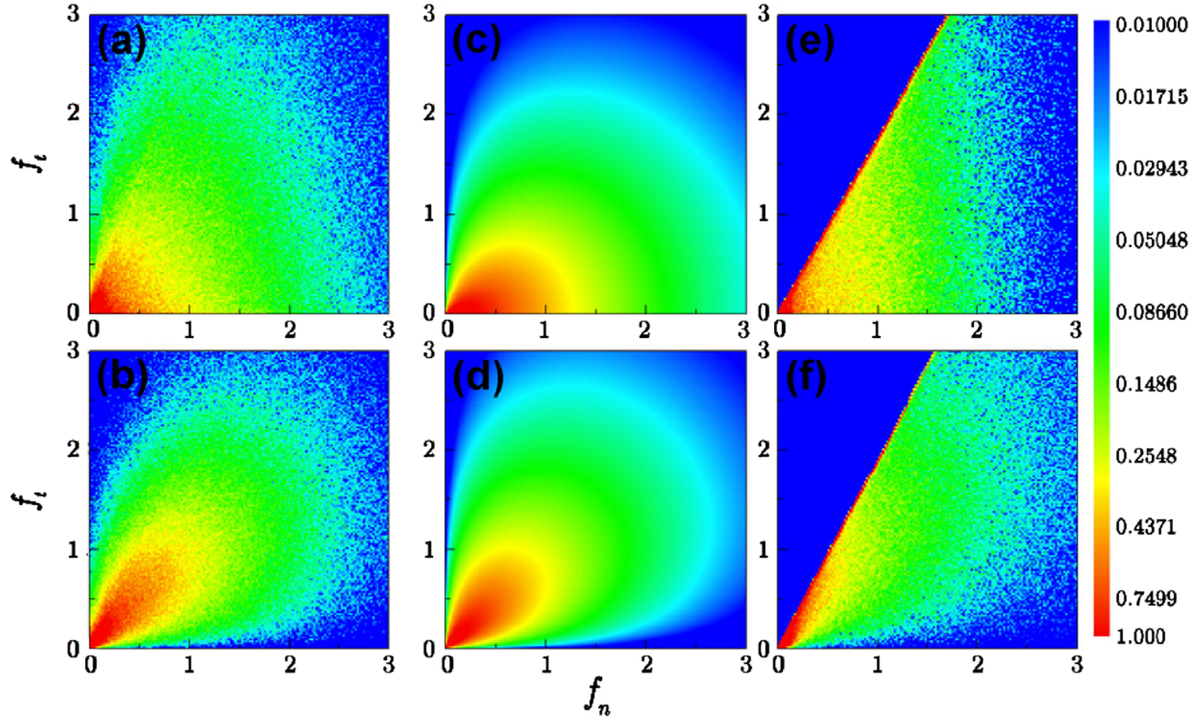


**Fig. 2.** (a), (b) Contour plots of $P_\infty(f_t, f_n)$ from simulation results in two dimensions and three dimensions, respectively; (c), (d) contour plots of the empirical formula Eq. (2) with $a = 0.8$ in two dimensions and three dimensions, respectively; (e), (f) contour plots of $P_{0.3}(f_t, f_n)$ from simulation results in two dimensions and three dimensions, respectively, with $\mu = 0.3$. In (a), (b), (e) and (f), we superpose the data from 20 individual configurations; each of them contains 10,000 grains.

with $\mu > 0.1$ for two dimensions in the inset of Fig. 1(a) due to the difficulty of preparing disordered two-dimensional monodisperse packing at small values of $\mu$.

(ii) We find that the contour plot of $P_\infty(f_t, f_n)$ follows the geometric behavior shown in Fig. 2(a) and (b), especially in the three-dimensional case where $P_\infty(f_t, f_n)$ is symmetric in the space of $(f_t, f_n)$. We will show later on that this symmetric behavior only occurs at large enough forces in three dimensions. A correct form of the force distribution should predict this behavior.

By fitting our numerical data, we find an empirical form of $P_\infty(f_t, f_n)$ for infinite friction, consistent with (i) and (ii). We describe it by defining new variables $f = \sqrt{f_t^2 + f_n^2}$ and $\theta = \arctan\left(\frac{f_t}{f_n}\right)$:

$$P_\infty(f, \theta) = a g(\theta) e^{-\sqrt{a} f}, \tag{2}$$

where $a$ is a constant, which could be regarded as the inverse of the angoricity [10–12]. By fitting this distribution, we find

$$g(\theta) = (D - 1)(\sin\theta)^{D-2} \cos\theta,$$

which can be regarded as the density of states approximately for the force ensemble at $\mu = \infty$. Eq. (2), plotted in Fig. 2(c) and (d), shows similar patterns to the simulation results of Fig. 2(a) and (b). We further study the contour plot of $P_\mu(f_t, f_n)$ at $\mu = 0.3$ shown in Fig. 2(e) and (f). $P_{0.3}(f_t, f_n)$ displays the same pattern inside the Coulomb cone as when $\mu = \infty$. We therefore suggest that the study of force distribution for frictional packing should focus on packings with $\mu = \infty$. The density of state $g(\theta)$ describes the probability of the contact forces for a single contact to have an angle $\theta$ [we note that there is no obvious geometric meaning for $\theta$, which is not the angle between the normal and the net contact force: $\theta = \arctan\left(\frac{f_t}{f_n}\right) = \arctan\left(\kappa \frac{F_t}{F_t}\right) \neq \arctan\left(\frac{F_t}{F_t}\right)$] and indicates that the normal and tangential forces are correlated to each other even when there is no Coulomb constraint.

We define $P_<(\theta)$ as the cumulative probability distribution of $\theta$ indicating the probability of the contact forces for a single contact to have an angle less than $\theta$, such that $g(\theta) = \frac{dP_<(\theta)}{d\theta}$. We find that

$$P_<(\theta) = (\sin\theta)^{D-1}. \tag{3}$$

This simple form $P_<$ could lead to a theoretical approach to the force distribution since it provides the density of states within the statistical mechanics framework [11,12].

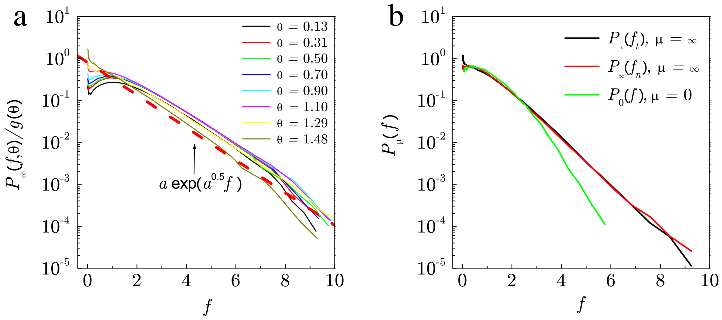


**Fig. 3.** (a) Log-linear plot of $P_\infty(f,\theta)/g(\theta)$ for various values of $\theta$ in three dimensions. All the curves collapse well with a pure exponential tail in the region of $f > 1$. The red dashed line is a function of $a\mathrm{e}^{-\sqrt{a}f}$ with $a = 0.8$. (b) Log-linear plot of $P_\infty(f_t)$, $P_\infty(f_n)$ and $P_0(f_n)$.

Eq. (2) implies that

$$P_\infty(f,\theta)/g(\theta) = a\mathrm{e}^{-\sqrt{a}f} \tag{4}$$

is independent of $\theta$. We plot $P_\infty(f,\theta)/g(\theta)$ for various values of $\theta$ in Fig. 3(a) to further compare with the simulation results. We find that all the curves collapse with exponential tails in the region of $f > 1$, indicating that the empirical form of Eq. (2) captures the main features of the force distribution for large forces. $P_\infty(f,\theta)/g(\theta)$ has a peak at $f \simeq 1$ when $\theta$ is small, and exhibits a monotonic exponential decrease when $\theta$ is close to $\frac{\pi}{2}$. This implies that the probabilities of single forces, $P_\infty(f_n)$ and $P_\infty(f_t)$, have different behavior, as shown in Fig. 3(b): $P_\infty(f_n)$ displays a peak at $f \simeq 1$ while $P_\infty(f_t)$ does not. This result is consistent with previous experimental studies of frictional packings [22]. In Fig. 3(b) we plot the force distribution at $\mu = 0$ and compare with that at $\mu = \infty$. We conclude that $P_0(f_n)$ has a stretched exponential tail that is close to Gaussian with a exponent $\beta = 1.65$ due to local entropy maximization [23].

By using Eqs. (1) and (2), we obtain a ratio force distribution $P_\infty(u)$, shown in Fig. 1 as a red dashed line, in good agreement with numerical results. This result further confirms that our empirical formula Eq. (2) is reasonable.

Further, we show that the ratio distribution is the link between the ensemble of forces and the average coordination number. We find that $P_\mu(u)$ can be rescaled to a single curve (except for the peak at $\mu$), with scaling factors equal to $\lambda$ and $\lambda^2$, for the $y$ and $x$ axes, respectively. We find $\lambda = 1$ in two dimensions and $\lambda = (z_c^0 - z_c^\infty)/(z_c^0 - z_c^\mu)$ in three dimensions, as plotted in Fig. 1(b). In the two-dimensional case, $P_\mu(u)$ collapses without scaling, so $\lambda = 1$. The three-dimensional case is different: we find that $\lambda \to 1$ when $\mu \to \infty$, so $P_\infty(u)$ does not change after multiplying the scaling factors. The factor $\lambda$ diverges at $\mu = 0$, implying that $P_\mu(u)$ reduces to a delta function at $\mu = 0$ due to the fact that all contact forces reach the Coulomb threshold in a pure frictionless packing.

Next, we show that the universal form of $P_\infty(u)$ determines $z_c^\mu$ for any $\mu$, hereby extending the isostatic counting argument from $\mu = 0$ and $\mu = \infty$ to finite values of $\mu$. From linear counting arguments we know that $z_c^0 = 2D$ and $z_c^\infty = D + 1$, and we want to interpolate to finite $\mu$ and obtain $z_c^\mu$. Below, we show that the Maxwell constraint arguments based on the number of redundant constraints provides the framework to derive $z_c^\mu$. Analysis of the coordination number of granular packings can be related to the Maxwell constraint counting in rigidity percolation theory [24]:

$$F = \frac{zD}{2}N - N_c + N_r, \tag{5}$$

where $F$ is the number of degrees of freedom (or floppy modes) satisfying $F \geq 0$, $N$ is the number of grains, $N_c$ is the number of constraints, $N_r$ is the number of redundant constraints, and $z$ is the coordination number. At the jamming transition, $F = 0$, resulting in a minimum value of $z$, i.e., $z_c^\mu$. Here $zDN/2$ is equal to the total number of unknown force variables for a fixed force network.

We consider a static packing with both force and torque balances, but without any typical constraints of translation and rotation. For packings with $\mu = \infty$, the number of constraints, $N_c$, will be equal to the number of force balance equations, $DN$, plus the number of torque balance equations, $D(D-1)N/2$, i.e., $N_c(\infty) = D(D+1)N/2$. There exists reasonable evidence [1,4,15–17,14,18–20] to believe that, at the jamming transition, $N_r(\infty) = 0$, implying a conjecture that the Maxwell counting approximation is exact. Therefore, $z = z_c^\infty = D + 1$. Another important case is at $\mu = 0$. Here the number of redundant constraints, $N_r(0) = D(D-1)N/2$, is equal to the number of torque balance equations due to the absence of tangential force. Further, we must add $z(D-1)N/2$ extra constraints to $N_c(0)$, corresponding to equations of tangential force equal to zero, ${F_t}^i = 0$. Therefore, $N_c(0) = N_c(\infty) + z(D-1)N/2$, and we obtain $z = z_c^0 = 2D$.

Analyzing intermediate values of $\mu$ is complicated, since many inequality constraints are created as $\mu {F_n}^i - {F_t}^i \geq 0$. Calculating $z_c^\mu$ becomes a nonlinear problem, which can be understood as an optimization of an outcome based on some set of constraints, i.e., minimizing a Hamiltonian of the system, $\mathbb{H}(\mathbf{F_n}, \mathbf{F_t})$, over a convex polyhedron specified by linear and non-negativity constraints. An interesting feature found in previous studies is that $z_c^\mu$ monotonically decreases from $2D$ to

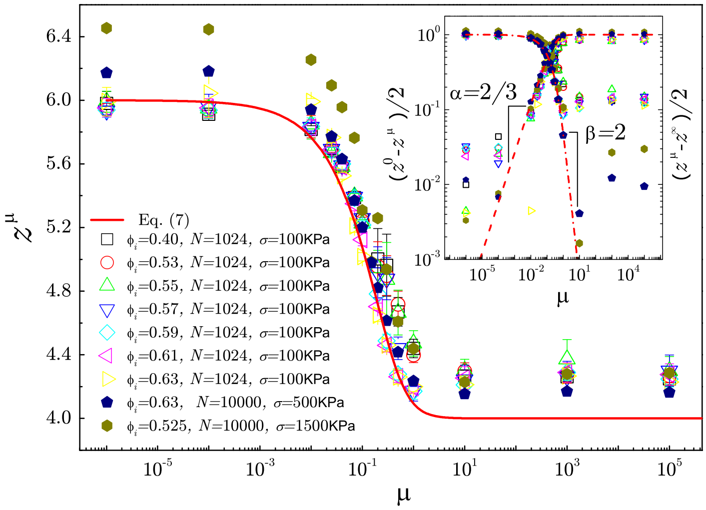


**Fig. 4.** $z^{\mu}$ versus $\mu$ for various values of the initial volume fraction $\phi_i$ in three dimensions. The red solid line is the theoretical result predicted by Eq. (8). In the inset, the red dashed and dashed–dotted lines are the prediction of $(z^0 - z^{\mu})/2$ and $(z^{\mu} - z^{\infty})/2$, respectively, in comparison with simulations.

$D + 1$ with increasing $\mu$ [4,16,19,20], implying that we can map this nonlinear problem to a linear one by considering a monotonic change in the number of constraints in Eq. (5) with increasing $\mu$.

The above analysis suggests extending the Maxwell counting argument Eq. (5) to a system with finite $\mu$ as

$$F = \frac{z_c^{\mu} D}{2} N - N_c(\infty) + \left[N_r(0) - z_c^{\mu}(D-1)N/2\right]\eta(\mu) = 0, \tag{6}$$

where $\eta(\mu)$ is an undetermined monotonic function ranging from 1 to 0 as $\mu$ ranges from 0 to $\infty$. The problem is reduced to choosing a functional form for $\eta(\mu)$.

To determine $\eta(\mu)$, we notice that it should be related to the sliding rate of packings, i.e., the ratio of the number of the sliding contacts to the number of total contacts in a packing, denoted $S(\mu)$. By definition, $S(\mu)$ is determined by $P_{\mu}(u)$, providing a link between the coordination number and the force distribution:

$$S(\mu) = 1 - \int_0^{\mu} P_{\mu}(u)\mathrm{d}u = 1 - \int_0^{\mu} \lambda^2 P_{\infty}(\lambda u)\mathrm{d}u. \tag{7}$$

The limiting cases are $S(0) = 1$ and $S(\infty) = 0$, and $S(\mu)$ has the same monotonic behavior as $\eta(\mu)$.

While $\eta(\mu)$ must be a function of $S(\mu)$, there are many choices for the functional relation between both quantities. We determine this functional form by fitting the simulations. Setting $\eta(\mu) = 1 - (1 - S(\mu))/\lambda$ provides very good fitting of $z_c^{\mu}$ with simulations in both two dimensions and three dimensions. Substituting $\eta(\mu)$ into Eq. (6), we arrive at a cubic equation for $z_c^{\mu}$ in three dimensions:

$$\frac{1}{\kappa^2\mu^2}\left(\frac{6 - z_c^{\mu}}{2}\right)^3 + 3\left(\frac{6 - z_c^{\mu}}{2}\right) - 3 = 0. \tag{8}$$

It can be shown that Eq. (8) predicts two power-law relations, $z_c^0 - z_c^{\mu} \sim \mu^{\alpha}$, and $z_c^{\mu} - z_c^{\infty} \sim \mu^{-\beta}$, respectively, for $\mu \to 0$ and $\mu \to \infty$, where $\alpha = 2/3$ and $\beta = 2$. In Fig. 4 we plot $z_c^{\mu}$ obtained from the cubic Eq. (8) and compare with simulation data in three dimensions. The asymptotic predictions of $\alpha = 2/3$ and $\beta = 2$ are in good agreement with the simulation results shown in the inset of Fig. 4. It is difficult to check the value of $\beta$ due to the difficulty of preparing a three-dimensional packing as close as possible to $z_c^{\infty} = 4$. To solve this problem, we prepare larger packings slightly above the critical point with a small constant pressure, and $z_c^{\mu}$ is replaced by $z^{\mu}$ without suffix. This result is shown in Fig. 4 with two sets of data for pressure $\sigma = 500$ kPa and $\sigma = 1500$ kPa. We can see that the power law of the coordination number is independent of the pressure even when $z^{\mu}$ is far from the isostatic value.

When we combine the power-law finding of $z_c^{\mu}$ with our theoretical work in [20] in three dimensions, where $z_c^{\mu}$ is linked to the volume fraction $\phi_c^{\mu}$ with a simple formula,

$$\phi_c^{\mu} = z_c^{\mu} \Big/ \left(z_c^{\mu} + 2\sqrt{3}\right), \tag{9}$$

then we solve the relation between $\phi_c^{\mu}$, $z_c^{\mu}$ and $\mu$; $\phi_c^{\mu}$ follows the same scaling behavior with $\mu$, $\phi_c^0 - \phi_c^{\mu} \sim \mu^{\alpha}$, and $\phi_c^{\mu} - \phi_c^{\infty} \sim \mu^{-\beta}$. Recent experiments [25] in three dimensions investigated the preparation of packings close to the random loose packing limit. They found $\alpha = 0.51 \pm 0.25$ and $\beta = 0.89 \pm 0.16$. Their measurement of $\beta$ is far away from our prediction, which could be due to the same problem that we find in our packings: the difficulty of preparing packings as close as possible to $z_c^\infty = 4$.

In the two-dimensional case, $\eta(\mu) = S(\mu)$ since $\lambda = 1$, and we have

$$z_c^\mu = \left[4 + \frac{2\kappa\mu}{(1+\kappa^2\mu^2)^{1/2}}\right] \Bigg/ \left[1 + \frac{\kappa\mu}{(1+\kappa^2\mu^2)^{1/2}}\right]. \tag{10}$$

This equation predicts $\alpha = 1$ and $\beta = 2$, close to our simulation result of $\beta = 1.86$. We cannot determine the value of $\alpha$ from simulation due to the difficulty of preparing disordered two-dimensional monodisperse packings for small values of $\mu$, and the polydispersity of packings may slightly affect these two indices. Previous simulations [19] of polydisperse two-dimensional packings have $\alpha = 0.7$, still close to our predictions.

In summary, we have developed a framework to study the connection between the force distribution and the coordination number. Some aspects of this connection remain empirical, including the density of states, $g(\theta)$, and the scaling factor $\lambda$, allowing for the collapse of $P_\mu(u)$ into a single curve. Overall, the obtained mathematical forms of the density of states, the different force distributions, the coordination number and the volume fraction may allow for their incorporation into a statistical force ensemble of jammed matter [11,12]. This may facilitate the solution of outstanding open problems such as the prediction of the power-law scaling of the pressure, the coordination number and elastic moduli with the volume fraction near the jamming transition [1].

## Acknowledgements

We acknowledge NSF-CMMT and DOE, Basic Sciences, Geosciences Division for financial support.

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