

Scaling Behavior of Driven Interfaces Above the Depinning Transition

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We study the depinning transition for models representative of each of the two universality classes of interface roughening with quenched disorder. For one of the universality classes, the roughness exponent changes value at the transition, while the dynamical exponent remains unchanged. We also find that the prefactor of the width scales with the driving force. We propose several scaling relations connecting the values of the exponents on both sides of the transition, and discuss some experimental results in light of these findings.

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Interface roughening in the presence of quenched disorder has recently been the focus of great interest [1]. In the typical case, a d -dimensional interface described by a height $h(\mathbf{x}, t)$ moves in a $(d + 1)$ -dimensional disordered medium. The randomness of the medium can be described by a quenched disorder $\eta(\mathbf{x}, h)$.

The continual motion of the interface in the medium requires the application of a driving force F . For driving forces smaller than a critical value F_c , the interface becomes pinned by the disorder after some finite time. For $F > F_c$, the interface moves with a constant velocity v . Hence, the velocity can be identified as the *order parameter* of the *depinning transition*. Close to the transition, the velocity of the interface scales as $v \sim f^\theta$, where θ is the velocity exponent, and $f \equiv (F - F_c)/F_c$ is the reduced force.

Experimental studies of the scaling of the *local* interface width, $w \sim \ell^\alpha$, where ℓ is the window of observation, reveal roughness exponents α in the range $0.6 - 1.0$ [2,3]. Several models in which quenched disorder plays an essential role have been proposed with the goal of explaining the experimental results (for reviews, see e.g. [1]). For these models, both the scaling of w with ℓ and the scaling of the *global* interface width W with the system size L , were studied.

A numerical study [4], recently confirmed by analytical arguments [5], revealed that most of the models introduced so far could be organized into *two* universality classes. A group of these models can be mapped, at the depinning transition, onto *directed percolation* (DP) for $d = 1$, or *directed surfaces* (DS) for $d > 1$ [6,7]. These models are referred to as *directed percolation depinning* (DPD) models. The mapping allows the determination of α from the correlation length exponents of DP or DS. Reference [4] showed that the DPD models can be described by a stochastic differential equation of the Kardar-Parisi-Zhang (KPZ) type [8] with quenched disorder [9]

$$\frac{\partial h}{\partial t} = F + \nabla^2 h + \lambda(\nabla h)^2 + \eta(\mathbf{x}, h), \quad (1)$$

where the coefficient λ of the nonlinear term diverges at

the depinning transition.

A number of different models, belonging to the second universality class, to which we refer to as *quenched Edwards-Wilkinson* (QEW), were found to have either $\lambda = 0$ or $\lambda \rightarrow 0$ at the depinning transition [4]. At the depinning, they can be described by the EW equation [10] with quenched disorder [11]

$$\frac{\partial h}{\partial t} = F + \nabla^2 h + \eta(\mathbf{x}, h). \quad (2)$$

This equation has been studied by means of the functional renormalization group [12]. Numerical studies of the pinned phase yield $\alpha \simeq 0.97$ [13] and $\alpha \simeq 1.25$ [14] in $(1 + 1)$ dimensions.

A remaining problem is the connection of these universality classes to the experiments. We observe that most numerical and analytical studies focus on the “pinned phase” ($F \leq F_c$) while nearly all experimental results are for the “moving phase” ($F > F_c$).

In this Letter, we study the moving phase in $(1 + 1)$ -dimensions for models representative of each of the two universality classes. For the DPD universality class, we find that both α and the growth exponent β — defined by $W \sim t^\beta$ for $t \ll t_\times(L)$, where $t_\times(L)$ is the saturation time — have different values on both sides of the depinning transition. However, we find that the dynamical exponent $z = \alpha/\beta$ has the *same* value in both phases.

First, we consider the QEW universality class. In $(1 + 1)$ -dimensions, we study a Hamiltonian model defined as

$$\mathcal{H} = \sum_{k=1}^L [(h_{k+1} - h_k)^2 - Fh_k + \eta(k, h_k)]. \quad (3)$$

Here, the first term represents the elastic energy that tends to smooth the interface, and $\eta(k, h_k)$ is an uncorrelated random number, which mimics a random potential due to the disorder of the medium. In the simulation, a column k is chosen and its height is updated to $(h_k + 1)$ if the change in (3) is negative. Thus, only motions that decrease the total energy of the system are accepted. Backward motions are neglected since these are rare events.

We start by studying the scaling of the local width w in a window of size ℓ , for different driving forces. In Fig. 1(a), we show the local width for the pinned and the moving phases. Consecutive slopes yield roughness exponents $\alpha_p \simeq 0.92$ for the pinned phase, and $\alpha_m \simeq 0.92$ for the moving phase (Table I). We also detect a dependence of the local width on L , that can be described by a scaling of the form [15]

$$w(\ell, L) \sim L^{\alpha_L} \Phi(\ell/L), \quad (4)$$

where $\Phi(u)$ is a scaling function that for $u \ll 1$ scales as u^{α_p} . We find $\alpha_L \simeq 1.23$.

We study the scaling of W with t , and find $\beta_p \simeq 0.85$ and $\beta_m \simeq 0.86$. These results imply that z remains unchanged at the transition.

The study of the scaling of the local width for the moving phase reveals the existence of two regimes. In the first

$$w \sim \ell^{\alpha_m} f^{-\chi_m} \quad [\ell \ll \xi], \quad (5)$$

where ξ is the correlation length, and χ_m is a new exponent that characterizes the dependence of the prefactor of the width on the driving force. In the second regime, the effect of the pinned disorder becomes irrelevant compared to the annealed disorder [16], and we obtain

$$w \sim \ell^{\alpha_a} f^{-\chi_a} \quad [\ell \gg \xi], \quad (6)$$

where χ_a is a new exponent, and α_a is the roughness exponent corresponding to annealed disorder. Depending on the absence or presence of nonlinear terms, we recover the results of either the EW or the KPZ equation with annealed disorder. This crossover was observed both for experiments and simulations of discrete models [1].

Since $\xi \sim f^{-\nu}$, where ν is the correlation length exponent, we propose the scaling ansatz

$$w(\ell, f) \sim \ell^{\alpha_a} f^{-\chi_a} g(\ell/\xi). \quad (7)$$

Upon comparison with (5) and (6), we find that the scaling function $g(u)$ satisfies $g(u \gg 1) = \text{const}$, and $g(u \ll 1) \sim u^{\alpha_m - \alpha_a}$. We also obtain

$$\chi_a = \chi_m + \nu(\alpha_m - \alpha_a). \quad (8)$$

In Fig. 2, we show the data collapse obtained using the results of Fig. 1(a) according to (7). The deviation from scaling for large values of ℓ/ξ is due to finite-size effects (see Fig. 3(b) for discussion).

To determine a second relation for the new exponent χ_m , let us consider the approach of the depinning transition for a system of size L . For such a system the transition does not occur for $f = 0$, as it would for an infinite system, but for an effective critical force such that $\xi \sim L$, implying $f \sim L^{-1/\nu}$. Replacing this result into (5), we obtain

$$w \sim \ell^{\alpha_m} L^{\chi_m/\nu} \quad [\ell \ll L]. \quad (9)$$

For the pinned phase we have, according to (4),

$$w \sim \ell^{\alpha_p} L^{\alpha_L - \alpha_p} \quad [\ell \ll L]. \quad (10)$$

At the transition, (9) and (10) must be identical. So, we find $\alpha_p = \alpha_m$, and

$$\chi_m = \nu(\alpha_L - \alpha_p). \quad (11)$$

Replacing the measured values of ν , α_L , α_p , and α_a into (8) and (11), we find $\chi_m \simeq 0.42$ and $\chi_a \simeq 1.04$, in good agreement with the values obtained in our simulations (Table I).

Let us now consider the DPD universality class. For a description of the model, we refer to the literature [6,7]. In Fig. 1(b), we show the local width for the pinned and the moving phases. We find $\alpha_p \simeq 0.63$ for the pinned phase, and $\alpha_m \simeq 0.75$ for the moving phase (Table I). While previous studies considered α_m as an effective exponent, and concluded that no self-affine scaling exists in the moving phase for $\ell \ll \xi$ [6,7], we find a consistent value for α_m regardless of how closely we approach the transition. This leads us to suggest the validity of the self-affine scaling for the moving phase. This conclusion is supported by the good data collapse obtained with (7), as shown in Fig. 3.

An argument similar to the one leading to (11) yields

$$\chi_m = \nu(\alpha_p - \alpha_m). \quad (12)$$

Replacing the known values of ν , α_p , α_m , and α_a into (8) and (12), we find $\chi_m \simeq -0.2$ and $\chi_a \simeq 0.23$. Although the agreement with the measured values is not exact, the error bars do not rule out the validity of (12).

For the DPD universality class, we also find that β changes values at the depinning transition (see Table I). We obtain for both sides of the transition $\alpha \simeq \beta$, implying that z remains unchanged at the transition [17]. The exponent z characterizes the time scale t_x for the propagation of correlations in the interface. This time scale is not expected to depend on the external force [19]. These results are in good agreement with a numerical integration of Eq. (1) [9].

Next, we compare our numerical results to earlier simulations and discuss their significance for the determination of the universality classes of several experiments. A problem with the interpretation of experimental and numerical results for the roughness exponent, has been the wide range of values for α : 0.5–1.0, measured in the moving phase. We note that, in this regime, the crossover to the annealed disorder regime leads to effective exponents that change with the velocity (or the driving force). For this reason, we suggest that the scaling function (7) might be useful in the determination of the exponents from the study of the local width w . As shown in Table I, the exponent χ_m has different signs for the two universality classes, leading to sharply distinct scaling behaviors for the prefactor of the width with the driving force. Since in many experiments it is possible to monitor the velocity of

the interface, and therefore the driving force, the study of this prefactor may also lead to an easier identification of the universality class to which the experimental results belong.

In light of this discussion, the interpretation of the results of Ref [20] is clear. The numerical integration of the EW equation with quenched disorder performed in Ref. [20] must belong to the QEW universality class, and the reported exponents are effective exponents whose values were affected by the crossover to the annealed regime.

The determination of the universality class of the *fluid invasion model* (FIM) of Ref. [21] is not trivial. However, we note that ν is identical for the FIM and for the Hamiltonian model. Furthermore, the value of α measured in Ref. [21] might be explained as an artifact of the use of least square fits to the local width.

Since the FIM is a realistic model for the fluid-fluid displacement experiments of Refs. [3], we propose that they belong to the universality class of Eq. (2). In fact, the range of roughness exponents measured in the experiments of Refs. [3] is consistent with the values that could be obtained with the Hamiltonian model. Also, in these experiments, it was found that the global width decreases with the increase of v , as for the QEW universality class.

The paper burning experiments of Ref. [2] and the imbibition experiments of Ref. [7] are believed to belong to the DPD universality class. This conclusion is supported by the values of the exponents measured in both experiments.

In summary, we find that for the DPD universality class, α and β change values at the depinning transition. We also find a new exponent χ_m , that relates the values of the roughness exponents on both sides of the transition. For the QEW universality class, α and β remain unchanged, and the exponent χ_m relates the different values of the local, α_p , and global, α_L , roughness exponents at the depinning transition.

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FIG. 1. (a) *QEW universality class*. Plot of the local width w as a function of ℓ for several values of the driving force. The results were averaged over 30 realizations of the disorder and the system size is 5524. (b) *DPD universality class*. Plot of the local width w as a function of ℓ for several values of the driving force. The system size is 10480 and each result was averaged over 50 realizations of the disorder. For clarity, the data for the pinned case is shifted by a factor of $1/2$.

FIG. 2. *QEW universality class*. Data collapse, according to (7), of the results displayed in Fig1(a).

FIG. 3. *DPD universality class.* (a) Data collapse according to (7), of the results displayed in Fig1(b). (b) Data collapse for $f = 0.0583$ and systems of different size. It is visually apparent that the deviation from scaling observed in (a) is due to finite size effects.

TABLE I. Critical exponents for the two universality classes studied in this paper. The subscript “ p ” refers to the pinned phase, the subscript “ m ” denotes the quenched disorder regime in the moving phase, and the subscript “ a ” refers to the annealed disorder regime. The exponents z and z_a were calculated from scaling relations, while the remaining exponents were calculated directly in the simulations. For the QEW, we obtain $\alpha_L = 1.23 \pm 0.04$.

Exponents	DPD	QEW
α_p	0.63 ± 0.03	0.92 ± 0.04
α_m	0.75 ± 0.04	0.92 ± 0.04
α_a	0.50 ± 0.04	0.46 ± 0.04
z	1.01 ± 0.10	1.45 ± 0.07
z_a	1.67 ± 0.26	2.09 ± 0.40
ν	1.73 ± 0.04	1.35 ± 0.04
θ	0.64 ± 0.12	0.24 ± 0.03
β_p	0.67 ± 0.05	0.85 ± 0.03
β_m	0.74 ± 0.06	0.86 ± 0.03
β_a	0.30 ± 0.04	0.22 ± 0.04
χ_m	-0.12 ± 0.06	0.44 ± 0.05
χ_a	0.34 ± 0.06	0.99 ± 0.05