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Modeling stratification in two-dimensional sandpiles

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Abstract

When a mixture of small and large grains is poured between two vertical slabs, the mixture spontaneously stratifies in alternating layers of small and large grains. We describe a possible mechanism and we develop a model that reproduces stratification of the large and small grains in alternating layers. We find that the requirement for stratification is that the angle of repose of the large pure species is larger than the angle of repose of the small pure species, a prediction confirmed by the experimental findings.

1. Introduction

Granular materials are known to exhibit unusual properties such as fluid and solid-like behavior, pattern formation and instabilities in flow through apertures, and size segregation [1,2]. Segregation of mixtures of grains of different size is commonly observed when granular materials are exposed to external periodic perturbations such as vibrations [3]. Recently, we showed [4] that periodic size segregation of grain mixtures can occur even in the absence of periodic external perturbations. When a mixture of grains of different sizes is poured between two vertical slabs with a gap of approximately 5 mm, a stratification of the mixture in alternating layers of small and large grains is observed. Additionally, there is an overall tendency for the large and small grain to *segregate* in different regions of the cell [4,5].

To describe the case of a single-species sandpile, in a two-dimensional geometry, Bouchaud et al. [6,7] developed a novel theoretical approach. They introduced two coupled variables: the local height of the sandpile, and the local density of rolling grains, and a set of coupled convection–diffusion equations to govern the flow of the rolling grains and their interaction with the sandpile. Recently, de Gennes [8]

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simplified the approach of Refs. [6,7], and very recently Boutreux and de Gennes (BdG) [9] treated the case of granular flows made of two species. BdG reproduced the phenomenon of *segregation*, but a theoretical understanding of *stratification* is lacking.

In this paper, we seek to understand segregation and stratification in the conditions of Ref. [4], where the two species have different size and different shape. We first introduce a discrete model to give a clear picture of the phenomenology, and then develop a continuum approach [10,12]. In agreement with the experimental findings [4], we find that segregation is related to the difference of size of the grains, and stratification, to the difference in repose angles of the two pure species. The first part of this paper is published in Ref. [11], where we discuss the experiments of stratification and segregation.

2. Discrete model

The sandpile is built on a lattice, where the grains have the same horizontal size as the lattice spacing and two different heights, H_1 and $H_2 > H_1$. Following Refs. [6–9], we regard each grain as belonging to one of two phases: a "static phase" (if the grain is part of the solid sandpile) and a "rolling phase" (if the grain is not part of the sandpile and rolls downward with a constant drift velocity). We consider the local slope $s_i \equiv h_i - h_{i+1}$ of the static grains as the variable controlling the dynamics of the rolling grains. Here h_i denotes the height of the sandpile at coordinate i.

At each time step a number of N_1 small grains plus N_2 large grains is deposited at the top of column 1 of the pile. These grains are considered as belonging to the rolling phase. One rolling grain of each species interacts with the surface at each time step, and can be converted from the rolling phase to the static phase. The remaining rolling grains are convected "downward" with unit "drift velocity" – i.e. they move to the next column at each time step.

The dynamics of each rolling grain interacting with the sandpile surface is governed by its local angle of repose (the maximum angle at which a rolling grain is converted into a static grain) [6,7].

We note that the repose angle depends on the local composition on the surface, so we define $\theta_{\alpha\beta}$ as the repose angle of a rolling grain of type α on a surface with local grains of type β. We choose $\theta_{21} < \theta_{12}$ to take into account that large grains roll more easily on top of small grains than small grains roll on top of large grains (since the surface "looks" smoother for large grains rolling on top of small grains. The repose angles of pure species $\theta_{\alpha\alpha}$ lie between θ_{21} and θ_{12}.

The stratification experiments [4] use a mixture of grains of different shapes (smaller "less faceted" grains and larger "more faceted" grains). The repose angle of the smaller pure species is then smaller than the repose angle of the large pure species – i.e., $\theta_{11} < \theta_{22}$. To mimic the experimental conditions for stratification [4], we set $\theta_{21} < \theta_{11} < \theta_{22} < \theta_{12}$.

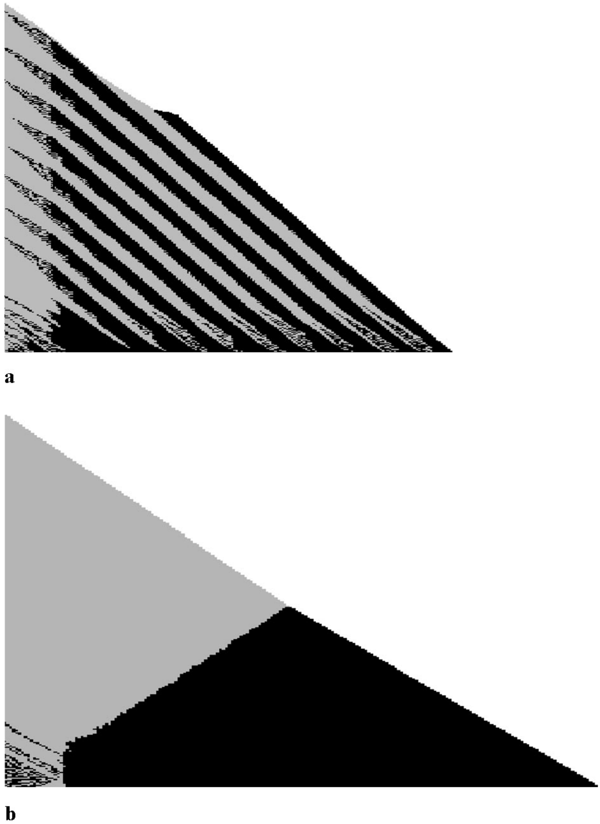

Fig. 1. (a) Stratification: Result obtained with the discrete model (for H_1=1, H_2=2, s_{11}=6, s_{12}=10, s_{21}=2, s_{22}=7, N_1=20, and N_2=10) reproducing the two main features of the experiment stratification, and segregation. Note the kink at which grains are stopped during an avalanche. The large grains are black and the small grains are gray. (b) Segregation: Result obtained with the discrete model when the angle of repose of the large grains is smaller than the angle of repose of the small grains (for H_1=1, H_2=2, s_{11}=6, s_{12}=10, s_{21}=2, s_{22}=5, N_1=20, and N_2=10).

At each time step, the rolling grain interacting with the sandpile surface at coordinate i will either stop – by being converted into a static grain – if the local slope of the surface $h_i - h_{i+1} \leqslant s_{\alpha\beta} \equiv \tan\,\theta_{\alpha\beta}$, or will continue to roll (together with the remaining rolling grains) to column $i+1$ if $h_i - h_{i+1} > s_{\alpha\beta}$. We iterate this algorithm to form a large sandpile of typically 10^5 grains.

Fig. 1a shows the resulting morphology. The stratification is qualitatively the same as that found experimentally [4], not only in regard to the *statics* of the sandpile (seen in Fig. 1a), but also in regard to the *dynamics*. If, on the other hand, we consider $\theta_{22} < \theta_{11}$ in the model (corresponding to a mixture of smaller "more faceted" grains, and larger "less faceted" grains) we find only segregation but not stratification (see Fig. 1b). Thus, the control parameter for stratification appears to be the difference in the repose angle of the pure species.

3. Theory for surface flow of granular mixtures

In a recent theoretical study for the case of a single-species sandpile Bouchaud et al. (BCRE) [6,7] have recognized the necessity of using two coupled variables to describe the avalanche dynamics of two-dimensional sandpile surfaces: the local angle of the sandpile $\theta(x,t)$ (or alternatively the height of the sandpile $h(x,t)$), and the local density of rolling grains $R(x,t)$. Here x is the longitudinal coordinate, and the pouring point is assumed to be at $x=0$. Recently, Boutreux and de Gennes (BdG) [9] has extended the BCRE formalism to the case of two species. This formalism considers the two heights of rolling grains $R_\alpha(x,t)$, with $\alpha=1,2$, respectively, for small and large grains, the height of the sandpile $h(x,t)$, and the volume fraction of static grains $\phi_\alpha(x,t)$ of type α at the surface of the pile. The equations proposed by BdG are [9]

$$\frac{\partial R_\alpha}{\partial t} = -v_\alpha \frac{\partial R_\alpha}{\partial x} + \sum_{\beta=1}^{2} M_{\alpha\beta} R_\beta \,, \tag{1}$$

$$\frac{\partial h}{\partial t} = -\sum_{\alpha\beta=1}^{2} M_{\alpha\beta} R_\beta \,. \tag{2}$$

Here, v_α is the downhill convection velocity of species α. The 2×2 collision matrix $M_{\alpha\beta}$ characterizes the interaction of the rolling grains with the surface, and it is determined by the local angle $\theta(x,t) \equiv -\partial h/\partial x$, the concentration $\phi_\alpha(x,t)$, and the angle of repose of the rolling grains [8]. The concentrations $\phi_\alpha(x,t)$ are given by

$$\phi_\alpha(x,t)\frac{\partial h}{\partial t} = -\sum_{\beta} M_{\alpha\beta} R_\beta \,, \tag{3}$$

and $\phi_1+\phi_2=1$. The steady-state solution of Eqs. (1)–(3) in the geometry of a silo has been found by BdG [9] in the case where the repose angle of each species does not depend on the composition of the surface, and using a specific form of the collision matrix M. This solution shows a complete segregation at the low edge of the silo.

4. Continuum model

Here, as in the discrete model, we focus on the dependence of the repose angle on the composition of the surface $\phi_\beta(x,t)$. The repose angle θ_α of each type of rolling grain is now a continuous and linear function of the composition of the surface $\theta_\alpha = \theta_\alpha(\phi_\beta)$. The repose angle $\theta_{\alpha\beta}$ defined for the discrete model is now $\theta_\alpha(\phi_\beta)$ with $\phi_\beta = 1$.

We propose that the elements $M_{\alpha\beta} \equiv M_{\alpha\beta}(R_\alpha, \phi_\beta, \theta)$ obeys the relation

$$\begin{aligned} M_{\alpha\alpha} &= \gamma\Pi[\theta(x,t)-\theta_\alpha(\phi_\beta)]\phi_\alpha - \gamma\Pi[\theta_\alpha(\phi_\beta)-\theta(x,t)]\,, \\ M_{\alpha\beta} &= 0\,, \end{aligned} \tag{4}$$

where $\Pi[x]=0$, if $x\leqslant 0$, and $\Pi[x]=x$, if $x\geqslant 0$. Here $v_\alpha/\gamma_{\alpha\beta}$ is the length scale at which a rolling grain will interact significantly with a surface at an angle of 1° above

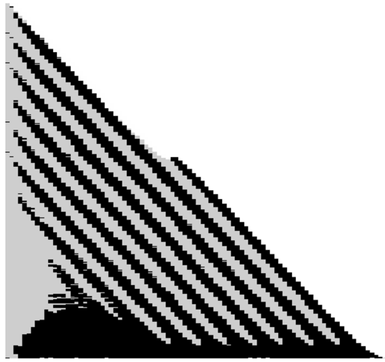

Fig. 2. Resulting morphology of the numerical integration of the continuum equations. Large grains are black and small grains are gray. The parameters used are $\tan(\theta_{11}) = 1$, $\tan(\theta_{22}) = 1.1$, $\tan(\theta_{12}) = 1.4$, $\tan(\theta_{21}) = 0.7$, $\gamma = 0.8$, and $v_1 = v_2 = 1$.

or below the angle of repose [8]. The collision matrix includes two type of processes. (a) *Capture*: when $\theta < \theta_\alpha$, and (b) *Amplification*: if $\theta > \theta_\alpha$.

We next solve Eqs. (1)–(4) numerically. The results, shown in Fig. 2, are qualitatively similar to the discrete model. We find stratification whenever $\theta_{22} > \theta_{11}$ (i.e., the repose angle for the pure species of large grains is larger than the repose angle for the pure species of small grains). We also find a "kink", corresponding to the growth of the new pair of static layers, with a well-defined steady-state profile and upward velocity. On the other hand we find only segregation when $\theta_{11} > \theta_{22}$.

4.1. Wavelength of the layers

The layer thickness λ is $R^0(v + v_\uparrow)/v_\uparrow$ (where $v_\uparrow$ is the uphill speed of the kink) which is a consequence of the conservation law stating that all the rolling grains are stopped at the kink. Furthermore, by dimensional analysis we find $v_\uparrow \propto \gamma R^0$, so that, for $R^0 \neq 0$, we obtain

$$\lambda \propto v/\gamma + R^0. \qquad (5)$$

5. Discussion

In sum, we develop a mechanism to explain the observed stratification [4]. This mechanism is related to the dependence of the local repose angle on the local surface composition. We find that stratification occurs only when the repose angle of the large grains is larger than the repose angle of the small grains ($\theta_{22} > \theta_{11}$, corresponding to large grains rougher than small grains). The model describes the static picture of the sandpile of Ref. [4] with alternating layers made of small and large grains, and also reproduces the dynamics, where the layers are built through a "kink" mechanism.

When $\theta_{22} < \theta_{11}$, the model predicts almost complete segregation, but not stratification. These results are in agreement with experiments [4].

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